Ethiopian Journal of Science and Sustainable Development

e-ISSN 2663-3205 Volume 11 (1), 2024

Journal Home Page: www.ejssd.astu.edu.et

EJSSD
ETHIOPIAN JOURNAL OF SCIENCE AND SUSTAINABLE DEVELOPMENT
An Official International Journal of Adama Science and Technology University

Research Paper

Finding the Values of Trigonometric Function without the Use of Scientific Calculators: The Case of Tangent Function

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Article Info

Article History:
Received 21 May 2023
Received in revised form 26 November 2023
Accepted 27 November 2023

Keywords:
comparison, derivation, existing values, formulaic values, tangent function

Abstract

In the realm of calculating trigonometric functions, computing the tangent function with a high degree of accuracy and precision without using scientific calculators has been an area of interest and importance. However, the manual determination of trigonometric values presents a formidable challenge, demanding careful attention to detail and the application of rigorous techniques. In this article, a set of formulas that enable the computation of tangent function values without resorting to scientific calculators are presented. The proposed formulas generate results that exhibit a remarkably high level of accuracy and precision, reproducible up to three decimal places. They are applicable within the degree intervals of 0 -21^o, 21-45^o and 45-90^o. For angles larger than 90 degrees, the unit circle definition of the function was utilized. The paper provides detailed and comprehensive mathematical derivations for each formula corresponding to their respective intervals and conducts a thorough comparison of the obtained results with the pre-existing tangent function values through the utilization of a tabular method of comparison. Considering the high degree of accuracy and precision exhibited by these formulas, their practical adoption by both students and teachers at all levels of academic pursuits are recommended. Overall, the formulas proposed have the potential to facilitate mathematical calculations and scientific research by providing more accurate and efficient means of calculating tangent function values.

1. Introduction

Trigonometric functions, as a fundamental branch of mathematics, comprise a group of six essential functions; namely, sine, cosine, tangent, cotangent, secant, and cosecant. Each function corresponds to distinct ratios of the sides of a right-angled triangle, providing precise relationships between angles and side lengths. These functions are also commonly known as circular functions, as they can be obtained as the ratios of the x and y coordinates of points on a circle with a radius of one, which correspond to angles in standard position.

The computation of trigonometric functions remains an essential task in modern engineering technology, as noted by Stewart et al. (2016), since these functions enable engineers to model, analyze, and design complex systems. In numerous engineering applications, engineers use trigonometry to determine the relationships between angles and forces, as well as to calculate the magnitudes of forces and other quantities of interest. According to the study conducted by Neill (2014), the computation of trigonometric functions is particularly important in fields such as aerospace engineering, where accurate calculations of angles and

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https://doi.org/10.20372/ejssdastu:v11.i1.2024.413

distances are crucial for designing aircraft and spacecraft. The precise computation of these functions is vital for the development of modern engineering technologies, encompassing diverse applications such as wireless communications, computer graphics, electrical power grids, and transportation systems.

However, many scholars acknowledge the task of manually calculating the values of trigonometric functions as a challenging one. According to Stewart (2016), this process involves the application of fundamental trigonometric identities and the ratios of sides of right-angled triangles. However, the complexity and number of these formulas often make even simple calculations time-consuming and error-prone. Tan (2014) argued that computing trigonometric functions by hand can be time-consuming and difficult, especially for non-right-angled triangles (p. 121). This is because trigonometric functions are operations that cannot be expressed as algebraic formulas involving arithmetical procedures, and students have trouble reasoning about such operations and viewing these operations as functions (Breidenbach et al., 1992), as well as the complex nature of the topic making it challenging for students to understand it conceptually (Demir, 2012). Besides, trigonometric functions are some of the functions students encounter that they are not able to evaluate directly through algebraic operations (Weber, 2008), and the functions are different from other forms of functions in that they cannot be computed directly by carrying out certain arithmetic calculations revealed by an algebraic formula. This demonstrates the complexity associated with manually calculating the values of trigonometric functions, and students have difficulty when asked to reason out about topics reliant on trigonometric function understanding (Brown, 2005; Thompson, 2008; Weber, 2005).

The values of trigonometric functions have been extensively tabulated and programmed into contemporary electronic calculators and computers, enabling the computation of trigonometric values with ease. However, the overreliance on calculators and other technology can have negative consequences. As students become more dependent on calculators and other technologies, they lose the ability to perform simple tasks and mathematical problems without such technologies (Mead, 2014), and these students are not competent without them. Without calculators, the students cannot pass the mathematics tests that the generation before them could (Lightner, 1999). Furthermore, a study by Bain (2015) shows that students use these devices to do all the work so that they do not need to learn concepts. Of course, students are allowed to not be as smart in mathematics because they have calculators to do the work for them (Jehlen, 2008).

The primary objective of this study is to introduce a conceptual framework and to comprehensively derive each formula within its respective interval. This involves elaborating on each formula and emphasizing its visualization and significance within the field of mathematical science. The subsequent section of the article focuses on comparing the computed results obtained using the proposed formulas with the values of the tangent function obtained from mathematical tools. Furthermore, the corresponding absolute differences between these values are calculated and tabulated to rigorously examine the study findings with precise mathematical accuracy.

2. Materials and Methods

The method used involved derivation of the formulas that incorporate the concept of an arithmetic sequence. It is important to note the inherent limitation that the tangent functions evaluated at integer angles do not exhibit an arithmetic progression. Nevertheless, this obstacle can be overcome by assuming that the tangent function values at integer angles are part of an arithmetic sequence. To derive the formulas, certain assumptions are made in each of the three intervals ($0 – 21^{o}$, $21 – 45^{o}$ and $45 – 90^{o}$). Although the domain for computing the tangent function values initially appears to be limited to the definition within right-angled triangles, the use of the unit circle definition allows for the extension of the function's domain to cover all real number degrees. In this context, the following formulas can be considered as the first, second, and third equations utilized for computing values within the respective intervals:

For the interval $0 – 21^{o}$, $\tan\theta = \frac{175\theta}{10^4 \times 1^\circ}$

For the interval $21 – 45^{o}$, $\tan\theta = \left(\frac{2d + 174^\circ}{10^4 \times 1^\circ \times 1^\circ}\right)\theta$

For the interval $45 – 90^{o}$, $\tan\theta = \frac{\ln\sqrt{2}\,(86^\circ + \theta)}{90^\circ - \theta}$

These formulas provide a mathematical representation for calculating the tangent values of angles within the specified intervals. To evaluate the values of the tangent function for any angle, the following guidelines were followed:

1) If the angle whose tangent function value is to be calculated is between 0 and 90°, the respective formula corresponding to the angle's interval is used.
2) If the angle whose tangent function value is to be calculated is greater than 90°, the angle's reference angle (θ_R) to determine the magnitude of the angle's tangent function value is used. Here, the reference angle (θ_R) for θ is the positive acute angle formed by the terminal side of an angle θ in standard position, whose terminal side does not lie on either of the coordinate axes.
3) The sign of the tangent function value is determined depending on the position of the angle's terminal side.

2.1 Derivation of the first formula

The first formula takes the form of the arithmetic progression formula, $A_n = A_1 + d(n-1)$, which is applied to discover the n^th^ term in an arithmetic sequence. In the context of the paper, the formula offers an avenue for calculating the values of tangent functions, effectively providing a solution to the problem of determining the unknown values. In this case, the formula under evaluation takes the form $\tan\theta = \tan 1° + d(\frac{\theta-1°}{1°})$, with θ defined in degrees. As with the arithmetic progression formula, this formula involves distinct components, namely $\tan\theta$ representing the nth term, $\tan 1°$ representing the first term, and *d* representing the common difference in the sequence. By adopting the value of 0.0175 for $\tan 1°$, it is possible to express the formula in terms of $\tan\theta = 0.0175 + \ d(\frac{\theta-1°}{1°})$. The question that arises is: from where does *d* originate and what value does it hold? The process of determining *d* involves steps outlined below.

Step 1: Identification of differences in tangent function values evaluated at successive integer angles ranging from 0 to 21°. The computed values include the difference between $\tan 1°$ and $\tan 0°$ which is 0.0175, the difference between $\tan 2°$ and $\tan 1°$ which is 0.0175, the difference between $\tan 3°$ and $\tan 2°$ which is 0.0175, up to $\tan 21°$ and $\tan 20°$ which is 0.0199.

Step 2: Obtaining the summation of differences computed in Step 1, which results in a total of 0.3839. This value coincides with the value of $\tan 21°$, which is the tangent value of the last integer angle in the interval.

Step 3: Dividing the total difference value obtained in Step 2 by the number of integers between 0 and 21, which is 22. This gives 0.0175, which is assigned to *d*. This value represents the effective common difference between successive tangent values of the angles over the interval of 0 to $21°$.

Step 4: Simplification of the formula for the tangent function. Using values obtained in Step 3, the general formula for the tangent function with a given angle θ of the interval 0 to 21° is given as:

$$\tan\theta = 0.0175 + 0.0175(\frac{\theta-1°}{1°})$$

$$= 0.0175\left(1+\left(\frac{\theta-1°}{1°}\right)\right)$$

$$= 0.0175\left(1+\left(\frac{\theta}{1°}\right)-1\right)$$

$$= \frac{175\left(\frac{\theta}{1°}\right)}{10^4}$$

Though the integral angles are used for the formula derivation, it works for any real angles (given in degrees) in the given interval of 0 to 21°.

2.2 Derivation of the second formula

The first formula for calculating the value of the tangent function appears to give the value with a high deviation from the true value of the function when angles exceed 21°. To address this discrepancy, adjustments were made to the formula by introducing additional variables, particularly in the numerator, as this approach has been found to increase the precision of the results. It is important to note, however, that the coefficient assigned to the variable *d* is determined only after numerous trials and errors. Here in the second formula, the variable *d* represents the degree difference between $21°$ and the angle whose value of the tangent function is to be computed. For example, given a desire to compute the tangent value of an angle of $35°$, the value of d will be $35° - 21° + 1°$ which equals $15°$. The specific equation utilized to generate the values of tangent function for any real angles between 21 and 45 degrees is expressed as:

$$\tan\theta = \left(\frac{2d + 174°}{10^4 \times 1°}\right)\left(\frac{\theta}{1°}\right)$$

2.3 Derivation of the third formula

According to Ratti et al. (2018), the tangent function is defined as the ratio of the sine and cosine functions of a given angle in a right-angled triangle. The utilization of this definition is fundamental in developing the third formula employed for determining the values of a tangent function for angles ranging from 45 to 90°. The formula used to find the value of the sine of angles from 60 to 90° is $\sin\theta = \frac{514+6(\frac{\theta}{1°})}{10^3}$ and the formula used to find the value of the cosine angle from 60 to 90° is $\cos\theta = \frac{173\frac{(90°-\theta)}{1°}}{10^4}$ (these formulas will be discussed in other articles in the future). Then the ratio will be evaluated as stated below:

$$\tan\theta = \frac{0.514 + \frac{0.006\theta}{1°}}{0.0173(90° - \theta)}$$

$$= \frac{(514+6\theta)10}{173(90° - \theta)}$$

$$= \frac{514+\frac{6\theta}{1°}}{17.3(90° - \theta)}$$

$$= 0.347\left(\frac{86° + \theta}{90° - \theta}\right)$$

$$= \ln\sqrt{2}\left(\frac{86°+\theta}{90°-\theta}\right) \text{ where } \ln\sqrt{2} = 0.347$$

Literally, the above ratio works just for angles ranging from 60 to 90°, as the corresponding formulas for sine and cosine are defined only in the mentioned interval. But, without losing generalization, it is also applicable for angles ranging from 45 – 60°, as the absolute difference between the formulaic result and the value from scientific calculators is less than 0.05, which is a moderately acceptable error in modern science measurement. Therefore, this formula is employed to find the values of the tangent function for angles ranging from 45 – 90°.

3. Results and Discussions

An effective method for determining whether formulas can accurately reproduce the values of the tangent function is through a comparison of results. This involves comparing the results from the formula with the calculator values of the function to assess the accuracy of the formula. By analyzing the absolute differences between the results, it is possible to determine how closely the formula can duplicate the values of the tangent function for angles. For a given evaluation, it is essential to carefully examine the values derived from the two sources, emphasizing any absolute differences or discrepancies that may be detected.

It is worth noting that the domain of the formula extends beyond integer angles and encompasses the real angles lying in the interval for the three proposed formulas. Additionally, it is better to recognize that the calculated dissimilarities are rounded off to 3 decimal places for these formulas.

3.1 The first formula

The differences between the formulaic results and the calculators' values is shown in Table 1. The acceptable error suggests that the formula accurately simulates the tangent function's extant values within the given range and is easily applied to manual computation.

3.2 The second formula

Table 2 provides a comparative analysis of the results obtained using the formula and the pre-existing values of the tangent function in the designated interval. The analysis reveals the deviation between the results generated by the formula and the existing values of the function, suggesting its high effectiveness in reproducing the values accurately over the given interval. The formula's user-friendliness is evident from its ability to be applied in computational settings by hand.

3.3 The third formula

From the data presented in Table 3, a slight deviation exists between the results of the two sides. This observation suggests that the formula is effective in reproducing the values of the tangent function within the specified interval and can be readily employed for manual computation.

Table 1: Comparison of results from the first formula and the pre-set values

Angle (°)	0	1	2	3	4	5	6	7	8	9	10
Pre-set values	0.000	0.018	0.035	0.052	0.070	0.088	0.105	0.123	0.141	0.158	0.176
Formulaic	0.000	0.018	0.035	0.052	0.070	0.088	0.105	0.123	0.141	0.158	0.175
Difference	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001

Angle (°)	11	12	13	14	15	16	17	18	19	20	21
Pre-set values	0.194	0.213	0.231	0.249	0.268	0.287	0.306	0.325	0.344	0.364	0.384
Formulaic	0.193	0.210	0.228	0.245	0.263	0.280	0.298	0.320	0.333	0.350	0.368
Difference	0.001	0.003	0.003	0.004	0.005	0.007	0.008	0.005	0.011	0.014	0.016

Table 2: Comparison of results from the second formula and the existing values

Angle (°)	21	22	23	24	25	26	27	28	29	30	31	32
Pre-set values	0.384	0.404	0.425	0.445	0.466	0.488	0.510	0.532	0.554	0.577	0.601	0.649
Formulaic	0.372	0.394	0.416	0.437	0.463	0.486	0.510	0.535	0.560	0.585	0.611	0.637
Difference	0.012	0.010	0.009	0.008	0.003	0.002	0.000	0.003	0.006	0.008	0.010	0.012

Angle (°)	33	34	35	36	37	38	39	40	41	42	43	44
Pre-set values	0.649	0.675	0.700	0.727	0.754	0.781	0.810	0.839	0.869	0.900	0.933	0.966
Formulaic	0.663	0.690	0.718	0.745	0.773	0.802	0.831	0.860	0.890	0.920	0.950	0.981
Difference	0.014	0.015	0.018	0.018	0.019	0.021	0.021	0.021	0.021	0.020	0.017	0.015

Table 3: Comparison of results from the third formula and the existing values

Angle (°)	45	47	49	51	53	55	57	59	61	63	65	67
Pre-set values	1.000	1.072	1.150	1.235	1.327	1.427	1.540	1.664	1.802	1.963	2.145	2.356
Formulaic	1.010	1.073	1.143	1.219	1.304	1.398	1.504	1.623	1.759	1.915	2.096	2.308
Difference	0.010	0.001	0.007	0.016	0.023	0.029	0.036	0.041	0.043	0.048	0.049	0.048

Angle (°)	69	71	73	75	77	79	81	83	85	87	89	90
Pre-set values	2.605	2.904	3.271	3.732	4.331	5.145	6.314	8.144	11.430	19.080	57.290	-
Formulaic	2.561	2.867	3.246	3.725	4.351	5.205	6.439	8.378	11.867	20.010	60.730	-
Difference	0.044	0.037	0.025	0.007	0.020	0.060	0.125	0.234	0.437	0.930	3.440	-

4. Conclusions and Recommendations

In this work, formulae that allow users to calculate the values of tangent functions without the need for scientific calculators have been presented. The key findings demonstrate that each formula in each corresponding interval is highly effective in reproducing the experienced values of the tangent function up to three decimal places. It is observed that the maximum absolute difference between the results obtained using the formulas and the values of the function programmed into a scientific calculator is less than 0.02, 0.03, and a yet-to-be-determined value for the first, second, and third formulae, respectively.

The increasing reliance of students on calculators and other technologies results in a loss of their capability to perform simple tasks and solve mathematical problems without the aid of such technologies. As these formulas are designed to be user-friendly and suitable for manual calculations, they are valuable in assisting users in learning concepts. Also, the approach promotes a greater understanding of the function's properties, and the results demonstrate that the formulae are effective in providing good estimates. Overall, the findings offer a unique perspective on computing the tangent function values, and it is expected that the study will contribute to the field of mathematics, specifically trigonometry.

Based on the effectiveness of the research findings in promoting a conceptual understanding of mathematics and encouraging manual calculations, two key courses of action are forwarded. Firstly, it is highly recommended and encouraged for other researchers to utilize this study as a fundamental framework for conducting further investigations aimed at enhancing the accuracy of the devised formulas in reproducing tangent function values. Future research can explore how effective these formulas are in different situations, considering different groups of learners. Secondly, it is strongly recommended that educational institutions integrate this new idea into their mathematics curricula. By incorporating these formulas into their teaching, educators will facilitate learners' development of a more comprehensive understanding of trigonometric functions and improve their manual calculation skills. Furthermore, by utilizing this new approach, students can acquire the necessary analytical and problem-solving skills to apply mathematics to real-world situations effectively. By doing so, it is possible to contribute to the development of more effective methods to teach and learn mathematics, ultimately advancing the understanding of trigonometric functions.

Acknowledgement: This work would not have been successful without the assistance of Adama Science and Technology University and the Oromia Development Association Special Boarding School. The author is sincerely grateful to both institutions for their invaluable support. Additionally, the author would like to express his appreciation to all the participants who contributed to the research. Finally, the author thanks the editors and peer reviewers for their insightful feedback on this work.

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